Parameter family of invertible spin chain systems

11/26/2025 . @ Kyoto

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§ Motivation

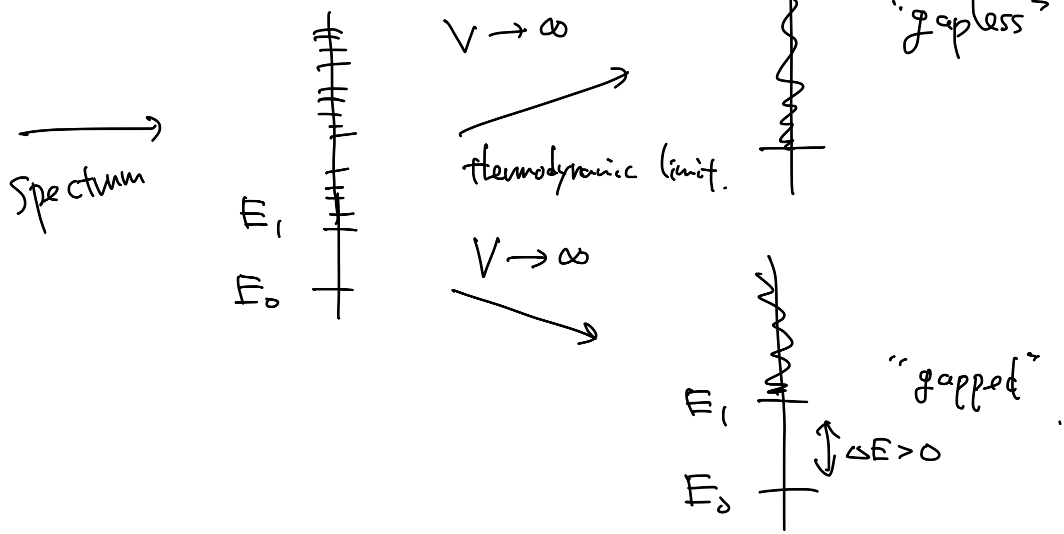
§ Matrix Product State

§ Parameter family and group action

§ Examples.

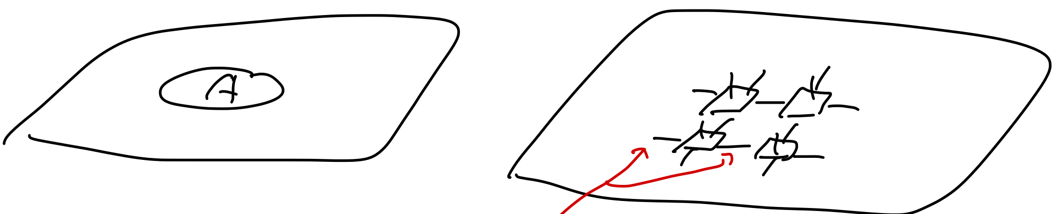
§ Motivation

H. Hamiltonian.



- Gapped state $\Rightarrow$ Area law. (conjecture)

$$S_{EE} = -\mathrm{Tr}\, \rho_A \log \rho_A = O(\partial A)$$

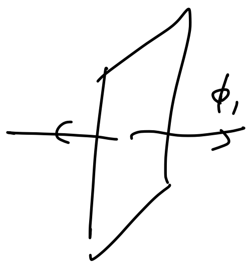


$\Rightarrow$ Approximated by tensor network state w/ finite bond dimension.

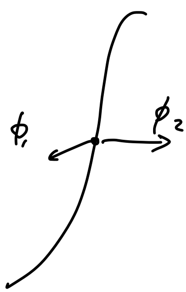
$\Rightarrow$ easy to study using linear alg.

I'm interested in higher codimensional defect in theory space.

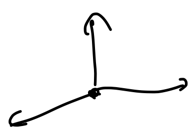
phase transition : codim-1 defect.



codim 2 defect.



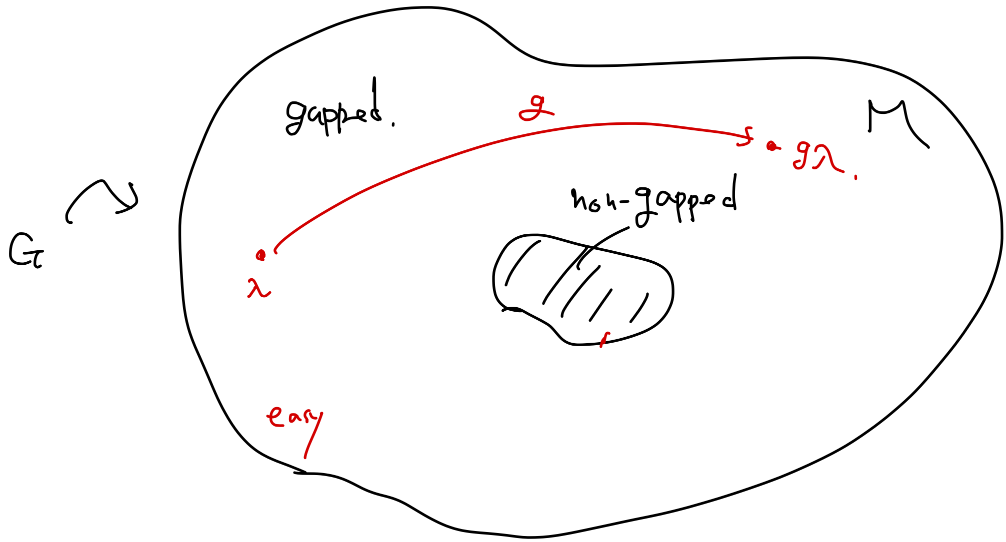
codim 3



Codim of defect = # of "orthogonal" relevant operators

What I want to discuss ...

$H(\lambda)$, $\lambda \in M$. parameter. family of Hamiltonian.



To give constraints on non-gapped region from gapped region enclosing it.

w/ Group (or category) action on prameter space.

$$U_g H(\lambda) = H(g\lambda) U_g \qquad g \in G.$$

Today's setup.

- 1D spin systems.
  - invertible state. (unique gapped state)
  - translational invariant (TI) $T|\psi\rangle = |\psi\rangle$.
  - G. finite group. $\curvearrowright$ parameter space M.

$$\tau \longmapsto g\tau \qquad g \in G, \tau \in M.$$

$$U_g H(\tau) U_g^{-1} = H(g\tau)$$

Goal

- TRS-LSM $\Rightarrow$ codim-3 defect.
- SPT phase transition $\Rightarrow$ codim-4 defect.

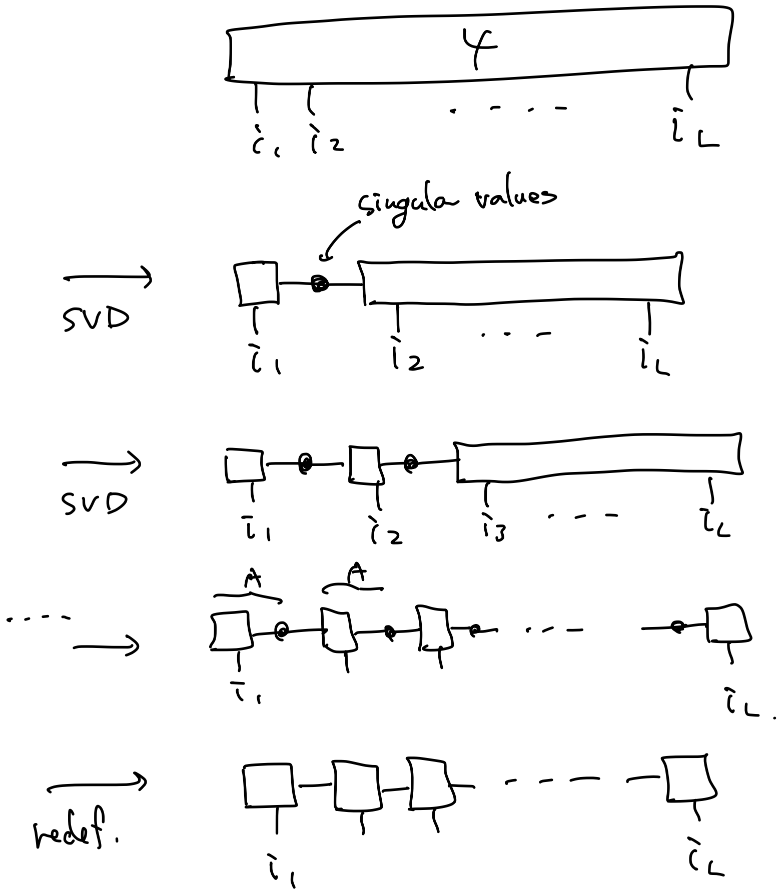
§ Matrix Product State (MPS)

[Perez-Garcia = Verstraete = Wolf = Cirac, quant-ph/0608197]

(my lecture note)

$$|\psi\rangle = \sum_{i_1\cdots i_L=1}^{d} \psi_{i_1\cdots i_L} |i_1\cdots i_L\rangle$$

$$\|$$



"Open MPS".

$$= A^{[1]i_1} \cdots\cdots A^{[L]i_L}.$$

site dep.

Fact.

$|\psi\rangle$ is TI. $\psi_{i_1 \cdots i_L} = \psi_{i_2 \cdots i_L i_1}$.

$\Rightarrow$ $\exists$ TI MPS.

$$|\psi\rangle = \sum_{i_1 \cdots i_L}^{d} \mathrm{tr}\left[A^{i_1} \cdots A^{i_L}\right] |i_1 \cdots i_L\rangle.$$

site indep.

$$=: |A\rangle.$$

$A^i \in M_D(\mathbb{C})$. D: bond dimension.

$$\begin{array}{ccc} \text{gapped TI state} & \xleftarrow{\text{approx.}} & \text{TI MPS.} \\ \cup & & \cup \\ \text{invertible TI state} & \xleftarrow[\text{approx}]{} & \text{normal MPS.} \end{array}$$

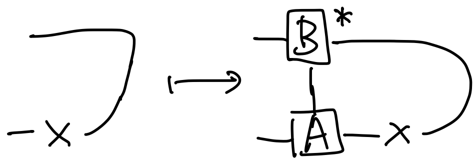
Def. (Transfer matrix)

A : MPS w/ bond dim. D

B : 〃 D'

$$T_A^B \in \mathrm{End}(M_{D\times D'}(\mathbb{C})),$$

$$T_A^B(X) := \sum_{i=1}^{d} A^i X B^{i\dagger}.$$



- $T \in \mathrm{End}(V)$. $\rho(T) = \max_{\mu \in \mathrm{Spec}(T)} |\mu|$. spectral radius.

- T is positive. $\Rightarrow$ $\rho(T) \in \mathrm{Spec}(T)$.

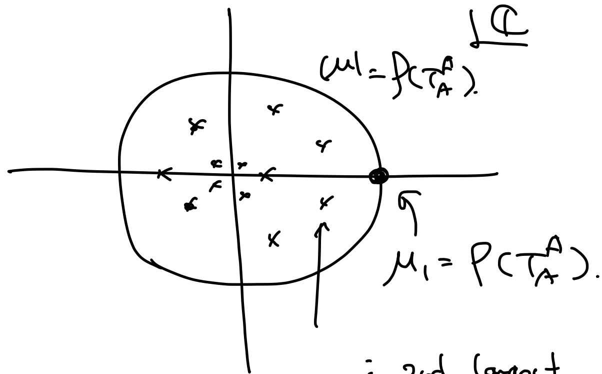
  $(X \geq 0 \Rightarrow T(X) \geq 0.)$ $\exists X \geq 0$, $T(X) = \rho(T) X$.

- T_A^A is positive.

Def. (normal MPS)

MPS A is normal

$\overset{\text{def.}}{\Longleftrightarrow}$
- $\mu = \rho(T_A^A)$ is non degenerate. (1-dim)
- Its eigenvector X is positive definite
- $\mu \in \mathrm{Spec}(T_A^A)$, $\mu \neq \rho(T_A^A) \Rightarrow |\mu| < \rho(T_A^A)$.



: 2nd largest

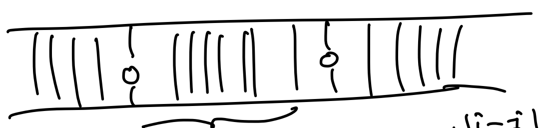
$|\mu_1| - |\mu_2| > \Delta\mu > 0$.
finite gap.

Fact.

MPS A is normal.

$$\Rightarrow \quad \langle O_i O_j' \rangle_{\mathrm{conn}} = \exp\left(-\frac{|i-j|}{\xi}\right).$$

$(\because)$ $\langle A | O_i O_j' | A \rangle$



$$(T_A^A)^l \longrightarrow \left|\frac{\mu_2}{\mu_1}\right|^{|i-j|}.$$

$$T_A^A = \mu_1 |X\rangle\langle Y| + \mu_2 |\ \rangle\langle\ | + \cdots$$

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"Gauge freedom" of normal MPS

A, B are MPSs related by

$$B^i = e^{i\theta} V^\dagger A^i V, \quad i = 1, \dots, d.$$

$\Rightarrow$ physically same state.

$$-\underset{|}{B}-\underset{|}{B}- \;=\; -V^\dagger-\overset{e^{i\theta}}{\underset{|}{A}}-V-V^\dagger-\overset{e^{i\theta}}{A}-V-\cdots$$

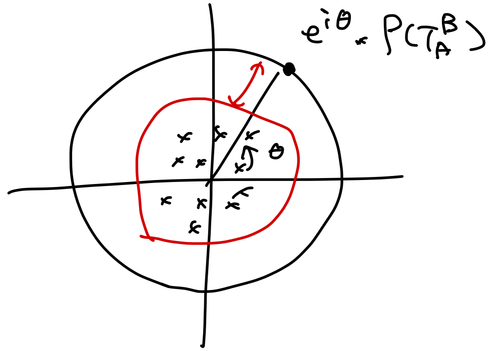
Theorem.

If A and B are normal.

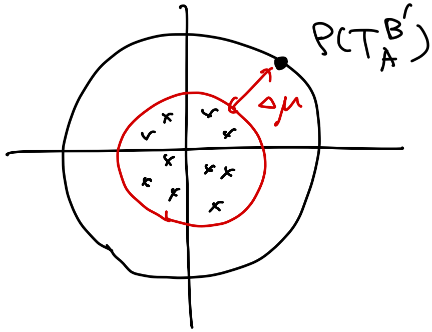
$\Leftarrow$ is also true.

- $A \underset{phys}{\sim} B$.

$\rightsquigarrow$ $\mathrm{Spec}(T_A^B) =$



$\Downarrow$ $B \mapsto B + \delta B = B'$



- $A \underset{close}{\sim} B \overset{def}{\iff} |\mu_1| - |\mu_2| > {}^{\exists}\Delta\mu > 0$.

$\rightarrow$ a kind of distance between two invertible TI states.

§ Parameter family and group action

[KS-Heinsdorf-Ohyama, 2305.08109 ; KS, 2507.19932]

- M. parameter space
- $A(\tau)$, $\tau \in M$. parameter family of normal MPSs.
- Introduce a fine triangulation of M ($\leftrightarrow$ simplicial str.)

(MPSs between all edges.

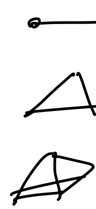
$A(\tau_0)$ $A(\tau_1)$ are close to each other.

τ_0 τ_1)

We can define.

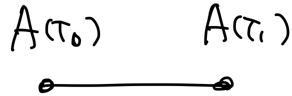
- 1-form $U(1)$ connection $a(\Delta^1)$
- 2-form $U(1)$ connection $\mathcal{A}(\Delta^2)$.

$\Delta^q = (\tau_0 \cdots \tau_q)$



Construction

On each edge $(\tau_0\tau_1)$



define a matrix $X(\tau_0\tau_1) \in \mathrm{Mat}_{D(\tau_0)\times D(\tau_1)}(\mathbb{C})$ by

$$T^{A(\tau_1)}_{A(\tau_0)}(X(\tau_0\tau_1)) = \sum_{i=1}^{d} A^i(\tau_0)\, X(\tau_0\tau_1)\, A^i(\tau_1)^{+}$$
$$= \mu_1(\tau_0\tau_1)\, X(\tau_0\tau_1)$$

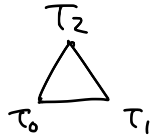
$$\rightsquigarrow \quad a(\tau_0\tau_1) := \mathrm{Arg}\left(\mu_1(\tau_0\tau_1)\right) \quad \in \mathbb{R}/2\pi\mathbb{Z}.$$

$$(da)(\tau_0\tau_1\tau_2) = a(\tau_1\tau_2) - a(\tau_0\tau_2) + a(\tau_0\tau_1) \sim 0.$$

$$\xmapsto{\text{lift}} f(\tau_0\tau_1\tau_2) \in \mathbb{R}. \quad \Rightarrow \quad \text{2-form curvature.}$$

$$ch_1 := \frac{1}{2\pi} \sum_{\Delta^2 \in \Sigma_2} f(\Delta^2) \in \mathbb{Z}. \quad \text{1st chern number.}$$

On each face $\Delta^2 = (\tau_0 \tau_1 \tau_2)$



2-form $U(1)$ connection $\mathcal{A}(\tau_0\tau_1\tau_2)$ is defined as.

$$\mathcal{A}(\Delta^2) := \mathrm{Arg}\left[\mathrm{Tr}\left[X(\tau_0\tau_1)\Lambda(\tau_1)^{\frac{2}{3}} X(\tau_1\tau_2)\Lambda(\tau_2)^{\frac{2}{3}} X(\tau_2\tau_0)\Lambda(\tau_0)^{\frac{2}{3}} \right]\right]$$

$\Lambda(\tau)$: Schmidt eigenvalues of A.

$\longrightarrow$ Wilson loop weighted by $\Lambda(\tau)$.

$$(d\mathcal{A})(\Delta^3) = 0 \in \mathbb{R}/2\pi\mathbb{Z}.$$

$\downarrow$ lift

$F(\Delta^3) \in \mathbb{R}$. 3-form curvature. higher Berry curvature.

$$N := \frac{1}{2\pi} \sum_{\Delta^3 \in \Sigma_3} F(\Delta^3) \in \mathbb{Z}.$$

Dixmier-Douady number.

With G-action

- G. finite group $\curvearrowright M$.

$$\tau \mapsto g\tau, \quad g \in G, \quad \tau \in M.$$

$$U_g H(\tau) U_g^{-1} = H(g\tau) \quad \text{———} \circledast$$

- $\phi : G \to \{\pm 1\}$. $\phi_g = \begin{cases} 1 & g \text{ is unitary} \\ -1 & g \text{ is anti unitary} \end{cases}$.

- Notation $\quad X^{\phi_g} = \begin{cases} X & \phi_g = 1 \\ X^* & \phi_g = -1. \end{cases}$

- $U_g = \bigotimes_x u_g$. $\quad u_g u_h^{\phi_g} = e^{i\omega_{g,h}} u_{gh},$

$$[\omega] \in H^2(G, \mathbb{R}/2\pi\mathbb{Z}).$$

- $$U_g |A(\tau)\rangle = \sum_{i_1 \cdots i_L} \mathrm{tr}[A^{i_1}_{(\tau)} \cdots A^{i_L}_{(\tau)}]^{\phi_g} \, U_g \, |i_1 \cdots i_L\rangle^{\phi_g}$$
$$= \sum_{i_1 \cdots i_L} \quad '' \quad \Big(\bigotimes_x \sum_{j_x} |j_x\rangle [u_g]_{j_x i_x} \Big).$$

$\rightsquigarrow$ G str. $\circledast$

$$\Longleftrightarrow \sum_j (u_g)_{ij} A^j(\tau)^{\phi_g} \underset{\text{phys.}}{\sim} A^i(g\tau). \quad \text{———} \circledast\circledast$$

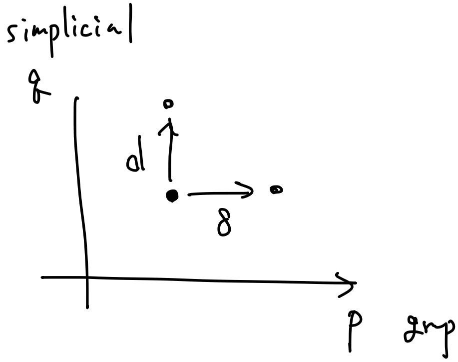
From this setup, we can define

1- and 2- cochains in G- simplicial complex

$$C^{p}_{group}\left(G,\ \underbrace{C^{q}_{simplicial}}_{\text{left. righ}}(M, \mathbb{R}/2\pi\mathbb{Z})\right).$$

with descendant equations.

$$\left(\begin{array}{ll} f(\Delta^q) \in C^q(M, \mathbb{R}/2\pi\mathbb{Z}). & \\ (g.f)(\Delta^q) = \phi_g f(\Delta^q). & \text{left action} \\ (f.g)(\Delta^q) = f(g\Delta^q) = f(g\tau_0, g\tau_1, \cdots, g\tau_q). & \\ & \text{right action.} \end{array}\right)$$



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$\underline{\delta}$

$f \in C^p_{grup}(G, A)$.

$(\delta f)_{g_1,\dots,g_{p+1}} = g_0 f_{g_2,g_3,\dots,g_{p+1}} - f_{g_1g_2,g_3,\dots,g_{p+1}} + (-1)^{j}\left(f_{g_1,\dots,g_p} \cdot g\right)$.

$\underline{d}$

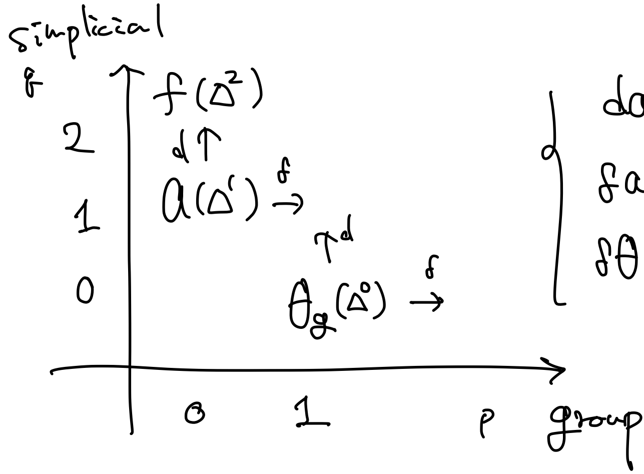
$f \in C^q(M, A)$.

$$df(\tau_0,\dots,\tau_{q+1}) = f(\tau_1\cdots\tau_{q+1}) - f(\tau_0\tau_2\cdots\tau_{q+1}) + \cdots + (-1)^{j} f(\tau_0\cdots\tau_q).$$

1-cochain

$$(a, \theta) \xrightarrow{D} \equiv f$$

$$(D = \delta + (-1)^p d)$$

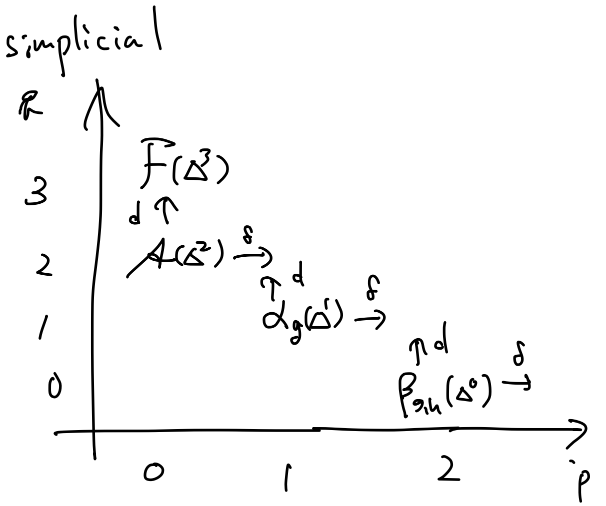


$$\left\{ \begin{array}{l} da \equiv f \mod 2\pi, \\ \delta a + d\theta = 0, \\ \delta\theta = \omega . \end{array} \right.$$

projective factor of u_g.

2-cochain

$$(A, \alpha, \beta), \xrightarrow{D} \equiv F.$$



$$\left\{ \begin{array}{l} dA \equiv F \mod 2\pi, \\ \delta A - d\alpha = 0, \\ \delta\alpha + d\beta = 0, \\ \delta\beta = 0 . \end{array} \right.$$

(Sketch of derivation)

(**)

(*)

$$\Rightarrow \sum_j (u_g)_{ij} A^j(\tau)^{\phi_g} = \underbrace{e^{i\theta_g(\tau)}}_{} V_g(\tau)^\dagger A^i(g\tau) V_g(\tau).$$

$\rightarrow$ $\theta_g(\Delta^0)$ def.

◎ $g \cdot h = gh$

$$\Rightarrow \quad e^{i\theta_g(h\tau)}\left(e^{i\theta_h(\tau)}\right)^{\phi_g} = e^{i\theta_{gh}(\tau)} e^{i\omega_{g,h}} \quad \Rightarrow \delta\theta = \omega.$$

$$\Rightarrow \quad V_g(h\tau) V_h(\tau)^{\phi_g} = \underbrace{e^{i\beta_{g,h}(\tau)}}_{} V_{gh}(\tau).$$

$\rightarrow$ $\beta_{g,h}(\tau)$ def.

◎ $(gh)k = g(hk)$ $\Rightarrow$ $\delta\beta = 0$.

(*) $\Rightarrow$ G - str. of transfer matrix.

$$\begin{array}{c} - V_g(\tau_1)^* - \\ \\ - V_g(\tau_0) - \end{array} \left(\begin{array}{c} - A(\tau_1)^* - \\ | \\ - A(\tau_0) - \end{array} \right)^{\phi_g} = e^{-id\theta_g(\tau_0\tau_1)} \begin{array}{c} - A(g\tau_1)^* - V_g(\tau_1)^* - \\ | \\ - A(g\tau_0) - V_g(\tau_0) - \end{array}$$

$\Rightarrow$ eigenvalue :

$$\mu_1(g\tau_0 g\tau_1) = e^{-i d\theta_g(\tau_0\tau_1)} \left(\mu_1(\tau_0\tau_1)\right)^{\phi_g}.$$

$$\xrightarrow{\text{Arg}} \quad \delta a - d\theta = 0.$$

eigenvec. :

$$V_g(\tau_0)\, X(\tau_0\tau_1)^{\phi_g}\, V_g(\tau_1)^{\dagger} = \underbrace{e^{i\alpha_g(\tau_0\tau_1)}}\, X(g\tau_0 g\tau_1).$$

$\longrightarrow$ $\alpha_g(\Delta^1)$ def.

◎ $g \cdot h = gh$ for $X(\tau_0\tau_1)$

$\Rightarrow$ relation betwen $e^{i\alpha_g}$ and $e^{i\beta_{g,h}}$.

$$\delta\alpha + d\beta = 0.$$

◎ G-action on higher Berry connection $\mathcal{A}(\tau_0\tau_1\tau_2)$

$\Rightarrow$ relation betwenn $\mathcal{A}(\tau_0\tau_1\tau_2)$ and $\alpha_g(\tau_0\tau_1)$.

$$\delta\mathcal{A} - d\alpha = 0.$$

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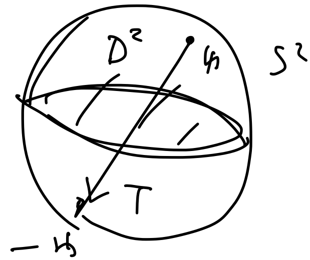
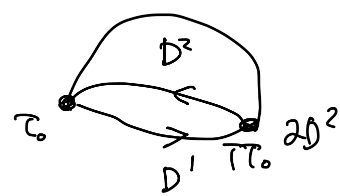
§ Applications.

Ex. 1 TRS- LSM anomaly. $u_T u_T^* = -1$.

$\mathbb{Z}_2^T$ action on 2- sphare S^2

$$n \longmapsto -n \in S^2.$$





$$\Rightarrow \quad ch_1 = \frac{1}{2\pi} \sum_{\Delta^2 \in S^2} f(\Delta^2)$$

$$= \frac{1}{2\pi} \times 2 \times \sum_{\Delta^2 \in D^2} f(\Delta^2)$$

$$\equiv_{\mathrm{mod}\,2} \frac{2}{2\pi} \times \sum_{\Delta^1 \in \partial D^2} a(\Delta^1)$$

$$= \frac{2}{2\pi} \times \sum_{\Delta^1 \in D^1} \big(\underbrace{a(\Delta^1) + a(T\Delta^1)}_{-(\delta a)_T(\Delta^1) \,=\, d\theta_T(\Delta^1).} \big)$$

$$= \frac{2}{2\pi} \times \sum_{\Delta^1 \in D^1} d\theta_T(\Delta^1)$$

$$= \frac{2}{2\pi} \times \big(\theta_T(\tau_0) - \theta_T(T\tau_0) \big)$$

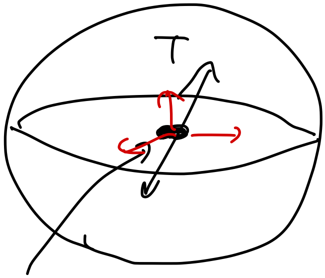
$$= \frac{2}{2\pi} \times (\delta\theta)_{T,T}(\tau_0)$$

$$= \frac{2}{2\pi} \times \underset{\pi}{\underset{\shortparallel}{\omega_{T,T}}} = 1 \quad \mathrm{mod}\ 2.$$

⇒ The TRS-LSM anomaly can be seen as the source of Berry curvature

2-form

param. space.



TRS-LSM pt

codim-3 defect.

in parameter space.

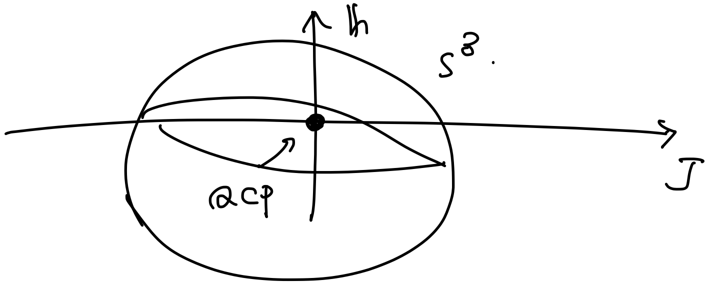
if TI is kept.

Ex. 2

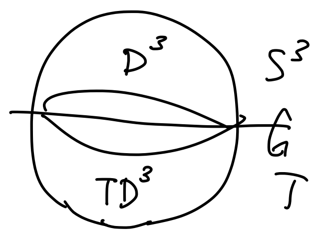
Phase transition between Haldane and triv. phases w/ TRS.

$$T H(J,h) T^{-1} = H(J,-h). \quad J \in \mathbb{R}, \; h \in \mathbb{R}^3.$$

$\Rightarrow$ $h=0$ is TRS-sym. line.



consider a S^3 surrounding QCP.



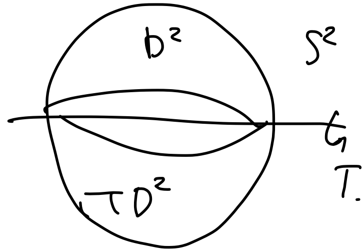
$$N = \frac{1}{2\pi} \sum_{\Delta^3 \in S^3} F(\Delta^3)$$

$$= \frac{2}{2\pi} \sum_{\Delta^3 \in D^3} F(\Delta^3)$$

$$\underset{\text{mod } 2}{\equiv} \frac{2}{2\pi} \sum_{\Delta^3 \in D^3} dA(\Delta^3)$$

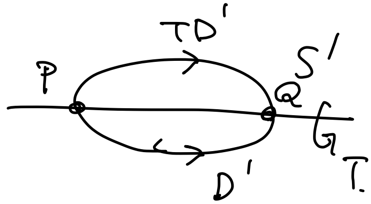
$$= \frac{2}{2\pi} \sum_{\Delta^2 \in \partial D^3} A(\Delta^2).$$

$$\partial D^3 = S^2.$$



$$\frac{2}{2\pi}\sum_{\Delta^2\in S^2}\mathcal{A}(\Delta^2)$$

$$= \frac{2}{2\pi}\sum_{\Delta^2\in D^2}\Big[\underbrace{\mathcal{A}(\Delta^2)+\mathcal{A}(T\Delta^2)}_{-(\delta\mathcal{A})_T(\Delta^2)\,=\,-(d\alpha)_T(\Delta^2).}\Big]$$



$$= \frac{2}{2\pi}\sum_{\Delta^2\in D^2}-(d\alpha)_T(\Delta^2)$$

$$= \frac{2}{2\pi}\sum_{\Delta^1\in \underset{\overset{\shortparallel}{S^1}}{\partial D^2}} -\alpha_T(\Delta^1).$$

$$= \frac{2}{2\pi}\sum_{\Delta^1\in D^1}\Big[\underbrace{-\alpha_T(\Delta^1)+\alpha_T(T\Delta^1)}_{-(\delta\alpha)_{T,T}(\Delta^1)\,=\,(d\beta)_{T,T}(\Delta^1)}\Big]$$

$$= \frac{2}{2\pi}\sum_{\Delta^1\in D^1}(d\beta)_{T,T}(\Delta^1)$$

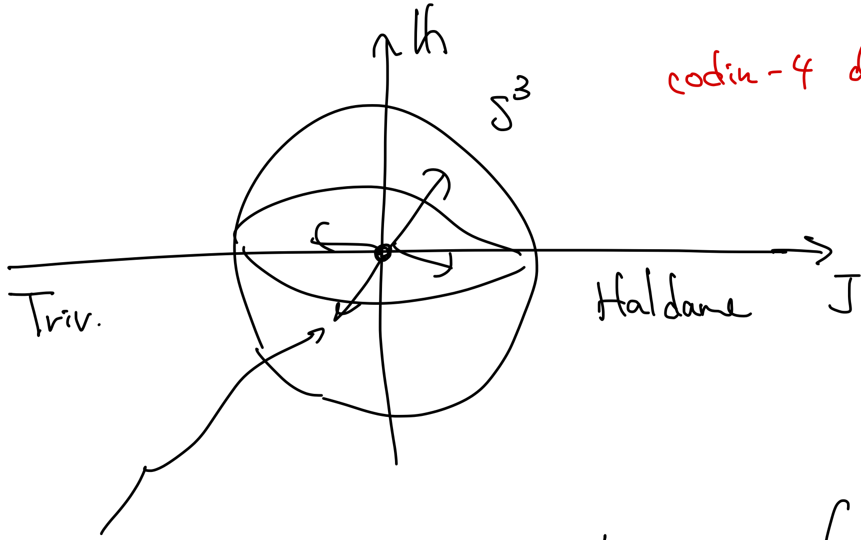
$$= \frac{2}{2\pi}\Big[\beta_{T,T}(P)-\beta_{T,T}(Q)\Big].$$

SPT invariant w/ TRS.

$$= \begin{cases} 0 & \text{same phase} \\ 1 & \text{diff. phase.}\end{cases}$$

$\Rightarrow$

codim-4 defect.



QCP can be seen as the source of 3-form higher Berry curvature.

back up.

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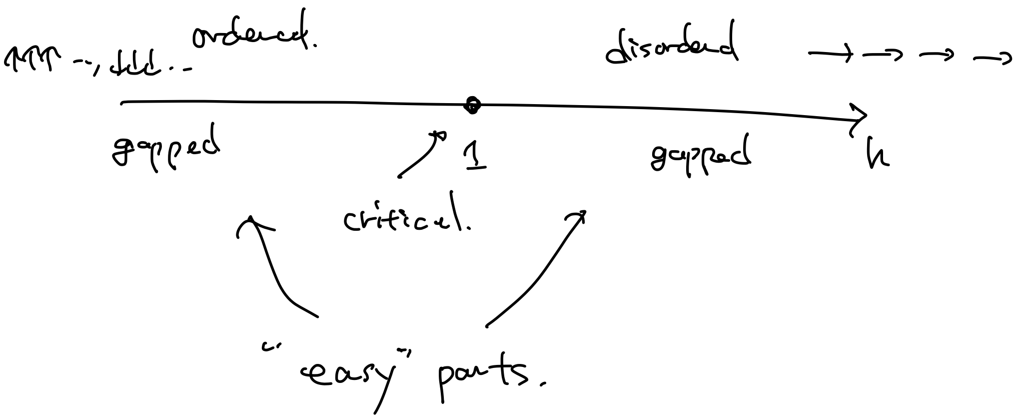
∘ Parameter family

e.g. 1D transverse Ising model

$$H = -\sum \sigma^z_j \sigma^z_{j+1} - h \sum \sigma^x_j .$$



· $\mathbb{Z}_2$ sym: $V = \prod \sigma^x_j$.

$$V\, H(h)\, V^{-1} = H(h) \longrightarrow$$ trivial action on parameter space

· Kramers-Wannier KW $\begin{cases} \sigma^x_j \longrightarrow \sigma^z_j \sigma^z_{j+1} \\ \sigma^z_j \sigma^z_{j+1} \longrightarrow \sigma^x_j \end{cases}$ nontriv action on param. space

$$\text{"} KW\, H(h) = H(h^{-1})\, KW \text{"}$$

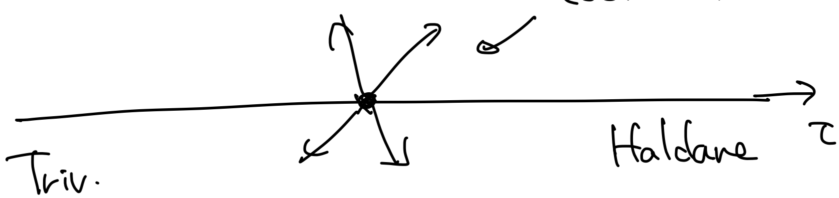
(not rigorous)

Goal

Skip

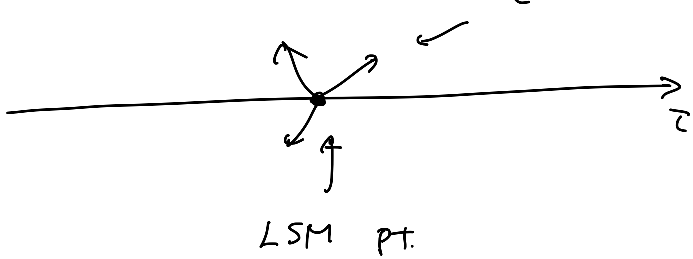
- The phase transition point between different 1D SPT phases is the source of the higher Berry curvature.

codim-4 defect.



- The LSM anomalous point is the source of Berry curvature.

codim-3 defect.



Area law

Skip

$$|\psi\rangle = \sum_\alpha \lambda_\alpha |L\alpha\rangle |R\alpha\rangle .$$

Schmidt decomposition

$$\rightarrow \rho_L = \mathrm{tr}_R |\psi\rangle\langle\psi| = \sum_\alpha \lambda_\alpha^2 |L\alpha\rangle\langle L\alpha| .$$

$$\rightarrow S_{EE} = -\mathrm{tr}\, \rho_L \log \rho_L .$$

$$= -\sum_\alpha \lambda_\alpha^2 \log \lambda_\alpha^2 .$$

[Arad-Kitaev-Landau-Vazirani, 1301.1162].

$$S_{EE} \le O\left(\frac{\log^3 d}{\Delta E} \right) .$$

d: dim of local Hilbert sp.

ΔE: spectral gap.

$$\Rightarrow -\sum_{\alpha=1}^{D} \lambda_\alpha^2 \log \lambda_\alpha^2$$

Prop. Skip.

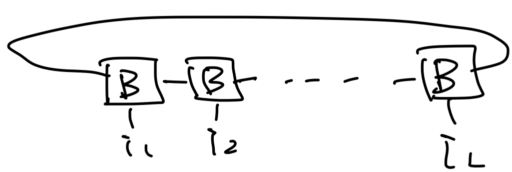
If $|\psi\rangle$ is translational invariant,

$\left(\Leftrightarrow \quad \psi_{i_1 \cdots i_L} = \psi_{i_2 \cdots i_L i_1} \right)$

then $\exists\, B \in M_{D\times D}(\mathbb{C})$, s.t.

$$\psi_{i_1 \cdots i_L} = \mathrm{Tr}\left[B^{i_1} \cdots B^{i_L} \right].$$

(=



).

(∵)

$$\psi_{i_1 \cdots i_L} = \mathrm{tr}\left[A^{[1+k]i_1} \cdots A^{[L+k]i_L} \right], \quad \forall k \in \mathbb{Z}_{\geq 0}$$

$$= \mathrm{tr}\left[B^{i_1} \cdots B^{i_L} \right],$$

$$B = L^{-\frac{1}{L}} \begin{bmatrix} 0 & A^{[1]} & & & \\ & 0 & A^{[2]} & & \\ & & 0 & \ddots & \\ & & & \ddots & A^{[L-1]} \\ A^{[L]} & & & & 0 \end{bmatrix}.$$

☆ translationally invariant state can be represented by MPS. //

Skip

Remark.

Using the gauge freedom,
normal MPSs can be chosen to satisfy.

$$
\begin{cases}
\sum_i A^i A^{i\dagger} = \mathbb{1}_D . \\
\sum_i A^{i\dagger} \Lambda^2 A^i = \Lambda^2 ,
\end{cases}
\qquad
\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_D \end{pmatrix},
$$

$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_D > 0$.

(Schmidt eigen values).

$\to$ " canonical condition "

$$
\Rightarrow \quad \rho(T_A^A) = 1 = \mu_{max}
$$

$$
|\mu_i| < 1 , \quad i \geq 2 .
$$

From now on. I always assume the canonical condition.

Remark.

Skip.

- A and B represent the same state.

$\Rightarrow$ $e^{-i\theta}$, V. are eigenvalue and eigenvector of mixed transfer matrix.

$$T_A^B = \sum_{i=1}^{d} A^i \otimes (B^i)^\dagger ,$$

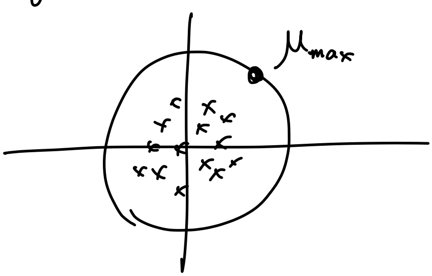
$$T_A^B(V) = e^{-i\theta} V .$$

$B^i = e^{i\theta} V^\dagger A^i V$

$$\because \quad T_A^B(V) = \sum_i A^i V B^{i\dagger}$$

$$= \sum_i e^{-i\theta} V \underbrace{B^i B^{i\dagger}} = e^{-i\theta} V . \quad //$$

- If MPSs A and B are physically "close" to each other, then the transfer matrix T_A^B also has unique largest eigenvalue in modulus.



$\because$ T_A^B depend continuously on MPS B.

$$T_A^B(X) = \sum_i A^i X B^{i\dagger} = \mu_{max} X .$$

$\rightsquigarrow$ X is well-def.